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Some Counterexamples Concerning Strong M-Bases of Banach Spaces

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A sequence of elements $(f_n)_1^\infty$ of a real or complex Banach space X is an M-basis (of X) if $\bigvee_{n=1}^\infty f_n = X$ and there exists a biorthogonal sequence of elements $(f_n^*)_1^\infty$ of X^* satisfying $\bigcap_{n=1}^\infty \ker f_n^* = (0)$. An M-basis $(f_n)_1^\infty$ is a strong M-basis if, additionally, $x \in \vee\{f_n : f_n^*(x) \neq 0\}$, for every element $x \in X$. Let X be a Banach space having a (Schauder) basis. We show that there exists a strong M-basis of X which is not finitely series summable. It follows that there is an atomic Boolean subspace lattice on X, with one-dimensional atoms, that fails to have the strong rank one density property. We show that there is always an atomic Boolean subspace lattice on X, with precisely four atoms, that also fails to have this density property. Also, if $X = c_0$ or c, an example is given of a strong M-basis $(f_n)_1^\infty$ of X such that $\bigvee_{n=1}^\infty f_n^* = X^*$ but with $(f_n^*)_1^\infty$ failing to be a strong M-basis of X^*. This partially follows from a description that is given of a class of strong M-bases of c_0, c, l^p $(1 \leq p < \infty)$. © 1994 Academic Press, Inc.

1. PRELIMINARIES

In [6] D. R. Larson and W. R. Wogen construct an interesting example of a reflexive operator T acting on complex separable Hilbert space with the property that the direct sum $T \oplus 0$ fails to be reflexive. This settles a long standing problem in operator theory.

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Amongst other things, the present paper elaborates on notions implicit in the basic construction of [6] in order to produce some counterexamples in the theory of bases and operator theory. Indeed, one of the counterexamples given below is outlined in a "Note added in proof" of [6] (see also [1, Addendum]) added after correspondence with the present authors. Here we give a slight generalisation of this counterexample, useful for other work included below. One of the main changes however is on the perspective and arguments. More precisely, in [6] the language and techniques are operator theoretic, using, for example, the identification of the set of bounded operators acting on a complex separable Hilbert space H with the dual of the ideal of trace class (nuclear) operators acting on H. In contrast, the methods used below are direct, use the language of the theory of bases and are valid for Banach spaces, real or complex, with a basis. Specifically, we begin by showing (Theorem 2.1) that on any Banach space with a (Schauder) basis there exists a strong M-basis which is not finitely series summable. By results of [1] this means that, on any such Banach space, there exists an atomic Boolean subspace lattice, hereafter abbreviated ABSL, with one-dimensional atoms which fails to have the strong rank one density property (this terminology is explained below). This result is the above-mentioned generalisation to the counterexample outlined in [6]. We then show (Theorem 3.1), again on any Banach space with a basis, that there exists an ABSL with four atoms that fails to have the strong rank one density property. This answers negatively, for ABSLs with finitely many atoms, a question raised in [1] (see also [8, 9]). Finally, we give an example (Theorem 4.2) of a strong M-basis $(f_n)_1^\infty$ of the Banach space $X = c_0$ or c, whose biorthogonal sequence $(f_n^*)_1^\infty$ satisfies $\bigvee_{n=1}^\infty f_n^* = X^*$ $(= l^1)$ but fails to be a strong M-basis of X^*.

For the most part, our notation and terminology will follow [1]. Throughout, the terms "Banach space," "subspace," and "operator" will mean "real or complex Banach space," "closed linear subspace," and "bounded linear operator," respectively. The dual of a Banach space X is denoted by X^* and $\hat{}$ denotes the canonical mapping of X into X^{**}. The set of operators acting on X is denoted by $\mathscr{B}(X)$. For any vector $f \in X$, $\langle f \rangle$ denotes the linear span of $\{f\}$. If $\{L_\gamma\}_\Gamma$ is a family of subspaces of X, $\bigvee_\Gamma L_\gamma$ denotes the closed linear span of $\bigcup_\Gamma L_\gamma$. For a family $\{f_\gamma\}_\Gamma$ of elements of X, $\bigvee_\Gamma \langle f_\gamma \rangle$ is denoted more simply as $\bigvee_\Gamma f_\gamma$. For any subset $\mathscr{E} \subseteq X$ the annihilator $\mathscr{E}^\perp$ of $\mathscr{E}$ is given by $\mathscr{E}^\perp = \{f^* \in X^* : f^*(x) = 0,$ for every element $x \in \mathscr{E}\}$. For every family $\{L_\gamma\}_\Gamma$ of subspaces X we have $(\bigcap_\Gamma L_\gamma)^\perp = \text{w}^* - \text{c.l.s.}\{\bigcup_\Gamma L_\gamma^\perp\}$, where "w* – c.l.s." means "weak* closed linear span of." If $f \in X$ and $e^* \in X^*$ the operator $e^* \otimes f$ acting on X is defined by $e^* \otimes f(x) = e^*(x) f$ $(x \in X)$.

A family $\mathscr{L}$ of subspaces of X is a *subspace lattice on* X if (0), $X \in \mathscr{L}$ and both $\bigvee_\Gamma L_\gamma$ and $\bigcap_\Gamma L_\gamma$ belong to $\mathscr{L}$ for every family $\{L_\gamma\}_\Gamma$ of elements of $\mathscr{L}$. For any subspace lattice $\mathscr{L}$, Alg $\mathscr{L}$ denotes the set of operators, acting on X, that leave every member of $\mathscr{L}$ invariant, that is,

$$\operatorname{Alg} \mathscr{L} = \{T \in \mathscr{B}(X): T(L) \subseteq L, \text{ for every } L \in \mathscr{L}\}.$$

An *atomic Boolean subspace lattice* (on X), abbreviated ABSL, is a subspace lattice $\mathscr{L}$ which is distributive and complemented (as a sublattice of the lattice of all subspaces of X) and has the property that every non-zero element of $\mathscr{L}$ contains an atom and is the closed linear span of the atoms it contains (an atom of $\mathscr{L}$ is a non-zero element which strictly contains no other non-zero element).

A subspace lattice $\mathscr{L}$ has the *strong rank one density property* if the algebra generated by the operators of Alg $\mathscr{L}$ of rank one, that is, the set of finite sums of the form ΣR_j, where each R_j is an operator of rank at most one belonging to Alg $\mathscr{L}$, is dense in Alg $\mathscr{L}$ in the strong operator topology. Since this algebra is a (two-sided) ideal of Alg $\mathscr{L}$, $\mathscr{L}$ has the strong rank one density property if and only if this algebra contains the identity operator on X in its strong closure. If $\mathscr{L}$ is an ABSL on X, the algebra generated by the rank one operators of Alg $\mathscr{L}$ coincides with the sub-algebra of finite rank operators of Alg $\mathscr{L}$ [3, 7].

A sequence $(f_n)_1^\infty$ of elements of X is *complete* if $\bigvee_{n=1}^\infty f_n = X$, and a sequence $(f_n^*)_1^\infty$ of elements of X^* is *total* if $\bigcap_{n=1}^\infty \ker f_n^* = (0)$. A complete sequence $(f_n)_1^\infty$ for which there exists a total (necessarily unique) biorthogonal sequence $(f_n^*)_1^\infty$ is called an *M-basis* of X. An M-basis $(f_n)_1^\infty$ of X is a *strong M-basis* of X if, additionally, $x \in \vee\{f_n: f_n^*(x) \neq 0\}$, for every element $x \in X$.

Below, several references are made to [1, Theorem 5.1]. The latter asserts the equivalence of several statements. We repeat here, for the reader's convenience, those statements that are of particular relevance here.

THEOREM [1, Theorem 5.1]. *Let $(f_n)_1^\infty$ be an M-basis of a Banach space X with biorthogonal sequence $(f_n^*)_1^\infty$. The following are equivalent.*

(1) *$(f_n)_1^\infty$ is a strong M-basis of X,*

(2) *$\{\langle f_n \rangle\}_1^\infty$ is the set of atoms of an ABSL on X,*

(3) *$\cap_I \ker f_n^* = \vee\{f_n: n \in \mathbb{Z}^+ \setminus I\}$, for every subset $I \subseteq \mathbb{Z}^+$,*

(4) *For every $\varepsilon > 0$ and every element $x \in X$ there exists a finite sum of the form*

$$F = \sum_{n=1}^{N} \lambda_n (f_n^* \otimes f_n) \qquad (\lambda_n \text{ scalars}) \text{ such that } \|Fx - x\| < \varepsilon.$$

In the above theorem, the equivalence of (3) and (4) was first proved in [10]. In the latter, sequences satisfying condition (4) are called 1-*series summable* and, more generally, an M-basis $(f_n)_1^\infty$ of X is called k-*series summable* (where $k \in \mathbb{Z}^+$) if, for every k-element set $x_1, x_2, \dots, x_k$ of elements of X and every $\varepsilon > 0$ there exists a finite sum F of the form as above such that $\|Fx_i - x_i\| < \varepsilon$, for $i = 1, 2, \dots, k$. Also in [10], an M-basis $(f_n)_1^\infty$ is called *finitely series summable* if it is k-series summable, for every $k \in \mathbb{Z}^+$. Let $(f_n)_1^\infty$ be a strong M-basis of X and let $\mathcal{L}$ be the ABSL on X having $\{\langle f_n \rangle\}_1^\infty$ as its set of atoms. Since the operator $R \in \mathscr{B}(X)$ is a rank one operator of Alg $\mathcal{L}$ if and only if $R = \lambda(f_n^* \otimes f_n)$, for some non-zero scalar λ and some $n \in \mathbb{Z}^+$ (see [1, p. 7]), it follows that $\mathcal{L}$ has the strong rank one density property precisely when $(f_n)_1^\infty$ is finitely series summable.

The usual (Schauder) basis of c_0, l^p $(1 \le p < \infty)$ is the sequence $(e_n)_1^\infty$ given by $e_n = (\delta_{mn})$. The corresponding biorthogonal sequence $(e_n^*)_1^\infty$ is given simply by $e_n^*(w) = w_n (w = (w_n))$. Here e_n^* corresponds to (δ_{mn}) under the usual linear isometry describing the dual space. The usual basis $(e_n)_1^\infty$ of c, on the other hand, is given by $e_1 = (1, 1, 1, \dots)$, $e_2 = (1, 0, 0, \dots)$, $e_3 = (0, 1, 0, 0, \dots), \dots$. In this case the biorthogonal sequence $(e_n^*)_1^\infty$ is given by $e_1^*(w) = \lim_{n \to \infty} w_n$, $e_2^*(w) = w_1 - \lim_{n \to \infty} w_n$, $e_3^*(w) = w_2 - \lim_{n \to \infty} w_n, \dots$ $(w = (w_n))$. The usual linear isometry of l^1 onto c^* is given by $(\alpha_n)_1^\infty \mapsto f^*$, where $f^*(w) = \sum_{k=1}^\infty \alpha_k w_{k-1}$, $w_0 = \lim_{n \to \infty} w_n$, and $w = (w_n)$. Under this isometry $e_1^*, e_2^*, e_3^*, e_4^*, \dots$ correspond to $(1, 0, 0, \dots), (-1, 1, 0, 0, \dots)$, $(-1, 0, 1, 0, 0, \dots)$, $(-1, 0, 0, 1, 0, 0, \dots) \dots$, respectively.

2. Failure of Finite Series Summability

A brief outline of a proof of a special case of the following result is given in [6]. Our proof is simpler and more direct; we explicitly produce the vectors that establish the failure of 2-series summability. By results of [1, p. 45 and p. 50] it follows that, on any Banach space with a basis there exists an ABSL with one-dimensional atoms that fails to have the strong rank one density property.

THEOREM 2.1. *On any Banach space with a basis there exists a strong M-basis which is not finitely series summable (in fact, not even 2-series summable).*

Proof. Let X be a Banach space with a basis $(e_n)_1^\infty$ consisting of unit vectors. Let $(e_n^*)_1^\infty$ be the sequence biorthogonal to $(e_n)_1^\infty$. Let $(b_n)_1^\infty$ be a sequence of non-zero scalars, converging to infinity fast enough so that

$$M = \sup_{n\geq 1}\left(|b_n|\sum_{k=n}^{\infty}\frac{1}{|b_k|}\right) < \infty.$$

(For example, we could take $b_n = r^n$, where $r > 1$.) Define the sequence $(f_n)_1^\infty$ of elements of X by

$$f_1 = e_1 + b_1 e_2, \qquad f_{2n-1} = -b_{n-1}e_{2n-2} + e_{2n-1} + b_n e_{2n} \qquad (n \geq 2),$$
$$f_{2n} = e_{2n} \qquad (n \geq 1).$$

We first show that $(f_n)_1^\infty$ is a strong M-basis of X.

Since for each $n \in \mathbb{Z}^+$ the linear spans of $\{f_i\colon 1 \leq i \leq 2n\}$ and $\{e_i\colon 1 \leq i \leq 2n\}$ are equal, the sequence $(f_n)_1^\infty$ is complete. It is readily checked that the sequence $(f_n^*)_1^\infty$ of elements of X^* defined by

$$f_{2n-1}^* = e_{2n-1}^* \qquad (n \geq 1),$$
$$f_{2n}^* = -b_n e_{2n-1}^* + e_{2n}^* + b_n e_{2n+1}^* \qquad (n \geq 1)$$

is total and biorthogonal to $(f_n)_1^\infty$. Thus $(f_n)_1^\infty$ is an M-basis. To show that $(f_n)_1^\infty$ is a strong M-basis, we must show, by [1, Theorem 5.1], that for every $\varepsilon > 0$ and every element $z \in X$ there is a finite linear combination of the $f_k^* \otimes f_k$ $(k \in \mathbb{Z}^+)$ whose value at z approximates z to within ε. Let $\varepsilon > 0$ and $z \in X$ be arbitrary. Let $z_n = e_n^*(z), n \geq 1$. Now, either $f_{2n}^*(z) \neq 0$ infinitely often or not. In the former case an easy calculation shows that

$$\left\|\left(\sum_{k=1}^{2n-1} f_k^* \otimes f_k + \frac{z_{2n} - b_n z_{2n-1}}{f_{2n}^*(z)} f_{2n}^* \otimes f_{2n}\right)z - z\right\| = \left\|\sum_{k=2n+1}^{\infty} z_k e_k\right\|$$

for every n for which $f_{2n}^*(z) \neq 0$. Since $\sum_{k=2n+1}^\infty z_k e_k \to 0$, the left hand side of the above is strictly less than ε for n sufficiently large.

Next, suppose that the latter case holds so that, for some $n_0 \in \mathbb{Z}^+$, we have $f_{2n}^*(z) = 0$, for every $n \geq n_0$. For such n $(\geq n_0)$ we have

$$z_{2n-1} = z_{2n+1} + z_{2n}/b_n,$$

and iteration shows that, for every $j \in \mathbb{Z}^+$, we have

$$z_{2n-1} = z_{2n+2j-1} + \sum_{k=n}^{n+j-1}\frac{z_{2k}}{b_k}.$$

Letting $j \to \infty$ and using the fact that $z_{2n+2j-1} \to 0$ we now see that the sum $\Sigma_{k=n}^{\infty} z_{2k}/b_k$ converges to z_{2n-1}, so

$$z_{2n-1} = \sum_{k=n}^{\infty} \frac{z_{2k}}{b_k} \qquad (n \geq n_0).$$

Hence

$$\begin{aligned} |b_n z_{2n-1}| &= |b_n| \left| \sum_{k=n}^{\infty} \frac{z_{2k}}{b_k} \right| \\ &\leq |b_n| \left(\sum_{k=n}^{\infty} \frac{1}{|b_k|} \right) \sup_{k \geq n} |z_{2k}| \\ &\leq M \sup_{k \geq n} |z_{2k}| \qquad (n \geq n_0). \end{aligned} \tag{1}$$

By choosing $n \geq n_0$ large enough, we can ensure that $M \sup_{k \geq n} |z_{2k}| < \varepsilon/2$ and at the same time $\|\Sigma_{k=2n}^{\infty} z_k e_k\| < \varepsilon/2$. For this n, using (1), we have

$$\begin{aligned} \left\| \left(\sum_{k=1}^{2n-1} f_i^* \otimes f_k \right) z - z \right\| &= \left\| b_n z_{2n-1} e_{2n} - \sum_{k=2n}^{\infty} z_k e_k \right\| \\ &\leq M \sup_{k \geq n} |z_{2k}| + \left\| \sum_{k=2n}^{\infty} z_k e_k \right\| \\ &< \varepsilon. \end{aligned}$$

This completes the proof that $(f_n)_1^{\infty}$ is a strong M-basis.

Next we show that, by an appropriate choice of $(b_k)_1^{\infty}$, it can be arranged that the strong M-basis $(f_n)_1^{\infty}$ fails to be 2-series summable. We explicitly produce two vectors x, y for which simultaneous pointwise approximation fails.

Choose $(b_k)_1^{\infty}$ so that (additionally) $b_k > 0$ $(k \in \mathbb{Z}^+)$ and the series $\Sigma_{k=1}^{\infty} 1/\sqrt{b_k}$ converges. (The choice $b_k = r^k$, with $r > 1$, mentioned earlier, already has this property.) With this choice both of the vectors

$$x = e_1 + \sum_{k=1}^{\infty} \frac{1}{\sqrt{b_{2k}}} (e_{4k-1} + e_{4k+1}),$$

$$y = \sum_{k=1}^{\infty} \frac{1}{\sqrt{b_{2k-1}}} (e_{4k-3} + e_{4k-1})$$

make sense (both series are absolutely convergent). We show that there is no sequence $(F_n)_1^{\infty}$ of finite linear combinations of the $f_k^* \otimes f_k$ $(k \in \mathbb{Z}^+)$ such that, simultaneously, $F_n x \to x$ and $F_n y \to y$ as $n \to \infty$. Suppose there was such a sequence $(F_n)_1^\infty$. Define elements $x^*, y^* \in X^*$ by

$$x^* = e_1^* - \sum_{k=1}^{\infty} \frac{1}{\sqrt{b_{2k}}} e_{4k}^*,$$

$$y^* = - \sum_{k=1}^{\infty} \frac{1}{\sqrt{b_{2k-1}}} e_{4k-2}^*.$$

(these vectors make sense; $(\|e_n^*\|)_1^\infty$ is uniformly bounded so both series converge absolutely). Calculations, which we omit here as more general ones can be found below in the proof of Theorem 3.1, show that

$$x^*((f_k^* \otimes f_k)x) + y^*((f_k^* \otimes f_k)y) = 0 \qquad (k \geq 1) \tag{2}$$

and so, by linearity,

$$x^*(F_n x) + y^*(F_n y) = 0 \qquad (n \geq 1).$$

Taking limits as $n \to \infty$ we obtain

$$x^*(x) + y^*(y) = 0.$$

But this contradicts the fact that $x^*(x) + y^*(y) = 1$ (this is readily verified; in fact $x^*(x) = 1$ and $y^*(y) = 0$). This contradiction completes the proof that $(f_n)_1^\infty$ is not 2-series summable.

Remarks. (1) In the special case of the preceding theorem whose proof is briefly outlined in [6] (and amplified in [1, Addendum]) $(e_n)_1^\infty$ is taken to be any orthonormal basis for the complex Hilbert space X and $b_n = 4^n$ $(n \geq 1)$.

(2) In the preceding theorem the condition that

$$\sup_{n \geq 1} \left(|b_n| \sum_{k=n}^{\infty} 1/|b_k| \right)$$

be finite, though sufficient, is not necessary for $(f_n)_1^\infty$ to be a strong M-basis of X. For instance, if $(b_n)_1^\infty$ is uniformly bounded, then from the equation

$$\left(\sum_{k=1}^{2n-1} f_k^* \otimes f_k \right) x - x = b_n x_{2n-1} e_{2n} - \sum_{k=2n}^{\infty} x_k e_k \qquad (n \geq 1)$$

and the fact that the right hand side of it converges to 0 as $n \to \infty$, it follows that $(f_n)_1^\infty$ is a strong M-basis.

(3) It is shown below (Theorem 4.1) that if X is any of c_0, c, l^p $(1 \le p < \infty)$ and $(e_n)_1^\infty$ is the usual basis, then $(f_n)_1^\infty$ as defined in the preceding theorem is a strong M-basis no matter what b_n we choose (the case where $X = l^2$ is established in [2]). In this case, the last half of the proof of Theorem 2.1 (beginning "Choose $(b_k)_1^\infty$ so that (additionally) $b_k > 0$ $(k \in \mathbb{Z}^+)$ and the series $\sum_{k=1}^\infty 1/\sqrt{b_k}$ converges ...") then shows that $(f_n)_1^\infty$ fails to be 2-series summable if $b_k > 0$ $(k \in \mathbb{Z}^+)$ and $\sum_{k=1}^\infty 1/\sqrt{b_k}$ converges. A slightly stronger result is proved in [2] for the case $X = l^2$: Given that $b_k > 0$ $(k \in \mathbb{Z}^+)$, $(f_n)_1^\infty$ fails to be 2-series summable if and only if $\sum_{k=1}^\infty 1/b_k$ converges.

3. Finite Atomic Boolean Lattices

The strong M-basis constructed in the proof of Theorem 2.1 is used below to settle in the negative an open question in the theory of non-selfadjoint operator algebras. As pointed out in the preceding section, this strong M-basis leads to an example of an ABSL with one-dimensional atoms that fails to have the strong rank one density property. The atoms in that case are simply the linear spans of the individual f_n $(n \ge 1)$. At the other extreme are ABSLs with finitely many (infinite-dimensional) atoms. It is shown in [1, Theorem 3.1] that every ABSL with two atoms has the strong rank one density property. In [4], a wide class of ABSLs with finitely many atoms also having this density property is exhibited. Nevertheless, the question: Must every ABSL with finitely many atoms have the strong rank one density property? remained open. Our next result answers this question in the negative.

Theorem 3.1. *On every Banach space with a basis there exists an ABSL with four atoms that fails to have the strong rank one density property.*

Proof. Let X be a Banach space with basis $(e_n)_1^\infty$ consisting of unit vectors and let $(e_n^*)_1^\infty, (f_n)_1^\infty, (f_n^*)_1^\infty$ be as defined in the proof of Theorem 2.1 with $(b_n)_1^\infty$ a sequence of positive scalars satisfying

$$\sup_{n \ge 1} \left(b_n \sum_{k=n}^\infty \frac{1}{b_k} \right) < \infty \qquad \text{and} \qquad \sum_{k=1}^\infty \frac{1}{\sqrt{b_k}} < \infty.$$

As proved in the preceding theorem, $(f_n)_1^\infty$ is a strong M-basis of X, so by [1, Theorem 5.1] the family of subspaces $\{\langle f_n \rangle\}_{n \in \mathbb{Z}^+}$ is the set of atoms of an ABSL on X. It follows (see [1, Example 2.7(ii)]) that the subspaces L_i

$(i = 1,2,3,4)$ defined by

$$L_i = \bigvee_{n=0}^{\infty} f_{4n+i} \qquad (i = 1,2,3,4)$$

are the atoms of an ABSL $\mathscr{L}$ on X. In this ABSL the (Boolean) complement L_i' of each L_i is the closed linear span of the other three atoms. Thus, for example,

$$L_1' = L_2 \vee L_3 \vee L_4 = \vee\{f_m : m \in \mathbb{Z}^+ \text{ not of the form } 4n + 1\}.$$

Hence, using [1, Theorem 5.1],

$$(L_1')^{\perp} = \left(\bigcap_{n=0}^{\infty} \ker f_{4n+1}^*\right)^{\perp} = \text{w}^* - \text{c.l.s.}\{f_{4n+1}^* : n \in \mathbb{N}\}$$

(where we have used the fact that $(\ker f_n^*)^{\perp} = \langle f_n^* \rangle, (n \geq 1)$). Similarly,

$$(L_i')^{\perp} = \text{w}^* - \text{c.l.s.}\{f_{4n+i}^* : n \in \mathbb{N}\} \qquad (i = 1,2,3,4).$$

We shall show that, for an appropriate choice of $(b_n)_1^{\infty}$,

$$x^*((e^* \otimes f)x) + y^*((e^* \otimes f)y) = 0 \tag{1}$$

for every rank one operator $e^* \otimes f$ of Alg $\mathscr{L}$, where the vectors x, y, x^*, y^* are defined as in the proof of Theorem 2.1. In the same way as in the latter proof, the fact that $x^*(x) + y^*(y) = 1$ shows that the identity operator does not belong to the strong operator closure of the algebra of finite rank operators of Alg $\mathscr{L}$.

To establish (1) we first show that, for every $i = 1,2,3,4$

$$x^*((f_{4m+i}^* \otimes f_{4n+i})x) + y^*((f_{4m+i}^* \otimes f_{4n+i})y) = 0 \qquad (m,n \in \mathbb{N})$$

that is,

$$x^*(f_{4n+i})f_{4m+i}^*(x) + y^*(f_{4n+i})f_{4m+i}^*(y) = 0 \qquad (m,n \in \mathbb{N}). \tag{2}$$

This is more general than Eq. (2) in the proof of Theorem 2.1 where we had $m = n$. The proof for this special case was omitted there because it is included in what we are now about to prove. For every $m, n \in \mathbb{N}$ we have, as can easily be verified

$$x^*(f_{4n+1})f_{4m+1}^*(x) = \sqrt{b_{2n}/b_{2m}} \qquad (\text{taking } b_0 = 1)$$

$$x^*(f_{4n+2})f_{4m+2}^*(x) = 0,$$

$$x^*(f_{4n+3})f_{4m+3}^*(x) = -\sqrt{b_{2n+2}/b_{2m+2}},$$

$$x^*(f_{4n+4})f_{4m+4}^*(x) = 0$$

and

$$y^*(f_{4n+1})f^*_{4m+1}(y) = -\sqrt{b_{2n+1}/b_{2m+1}},$$
$$y^*(f_{4n+2})f^*_{4m+2}(y) = 0,$$
$$y^*(f_{4n+3})f^*_{4m+3}(y) = \sqrt{b_{2n+1}/b_{2m+1}},$$
$$y^*(f_{4n+4})f^*_{4m+4}(y) = 0.$$

Hence the left hand side of Eq. (2) takes the values

$$\sqrt{\frac{b_{2n}}{b_{2m}}} - \sqrt{\frac{b_{2n+1}}{b_{2m+1}}}, \qquad 0, \qquad -\sqrt{\frac{b_{2n+2}}{b_{2m+2}}} + \sqrt{\frac{b_{2n+1}}{b_{2m+1}}}, \qquad 0$$

for $i = 1, 2, 3, 4$ respectively. Obviously these values are all zero in the case $m = n$ needed in the proof of Theorem 2.1. But they are also all zero if $b_n = r^n$ with $r > 1$ and we will require the b_n ($n \in \mathbb{Z}^+$) to be of this form in the remainder of this proof. By linearity, Eq. (1) holds for every element $e^* \in$ l.s. $\{f^*_{4n+i}\colon n \in \mathbb{N}\}$ and every element $f \in$ l.s. $\{f_{4n+i}\colon n \in \mathbb{N}\}$, for every $i = 1, 2, 3, 4$ (where "l.s." means "linear span of"). A simple argument now shows that Eq. (1) holds for every element $e^* \in$ w* − c.l.s.$\{f^*_{4n+i}\colon n \in \mathbb{N}\} = (L'_i)^{\perp}$ and every element $f \in \bigvee_{n=0}^{\infty} f_{4n+i} = L_i$, for every $i = 1, 2, 3, 4$. But, if the rank one operator $e^* \otimes f$ belongs to Alg $\mathscr{L}$, then $f \in L_i$ and $e^* \in (L'_i)^{\perp}$ for some i (see [1, p. 7]). The proof is therefore complete.

Remarks. (1) The proof of the preceding theorem shows that the ABSL $\mathscr{L}$ even fails to have the following "2-density" property: For every elements $x, y \in X$ and every $\varepsilon > 0$ there exists a finite rank operator $F \in$ Alg $\mathscr{L}$ such that $\|Fx - x\| < \varepsilon$ and $\|Fy - y\| < \varepsilon$. Using this counterexample we can make the observation, similar to that made in [2] in an analogous situation, that, for an ABSL with finitely many atoms, the algebra generated by the identity operator and the finite rank operators in its Alg need not be "2-dense" in its Alg, in the obvious sense (so need not be strongly dense). One simply has to consider, on $X \oplus X$, the ABSL $\mathscr{L}^{(2)}$ with atoms $L_i \oplus L_i$ ($i = 1, 2, 3, 4$), where $\mathscr{L}$ is any ABSL on X, with atoms L_i ($i = 1, 2, 3, 4$), that fails to have the "2-density" property described above. Let $x, y \in X$ be vectors that show the failure of "2-density" for $\mathscr{L}$ and let $T \in \mathscr{B}(X \oplus X)$ be the operator given by $T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. We claim that the following is false: For every $\varepsilon > 0$ there exists a scalar λ and a finite rank operator $F \in$ Alg $\mathscr{L}^{(2)}$ such that $\|(\lambda I + F)(x \oplus x) - T(x \oplus x)\| < \varepsilon$ and $\|(\lambda I + F)(y \oplus y) - T(y \oplus y)\| < \varepsilon$. For, otherwise, there exists sequences $(F_j^{(n)})_1^{\infty}$ ($j = 1, 2, 3, 4$) of finite rank operators of Alg $\mathscr{L}$ and a sequence $(\lambda_n)_1^\infty$ of scalars such that, as $n \to \infty$

$$\left[\lambda_n\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + \begin{pmatrix} F_1^{(n)} & F_2^{(n)} \\ F_3^{(n)} & F_4^{(n)}\end{pmatrix}\right]\begin{pmatrix} x \\ x\end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} x \\ x\end{pmatrix}$$

and

$$\left[\lambda_n\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + \begin{pmatrix} F_1^{(n)} & F_2^{(n)} \\ F_3^{(n)} & F_4^{(n)}\end{pmatrix}\right]\begin{pmatrix} y \\ y\end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} y \\ y\end{pmatrix}.$$

But this gives

$$\lambda_n x + \left(F_1^{(n)} + F_2^{(n)}\right)x \to x,$$
$$\lambda_n x + \left(F_3^{(n)} + F_4^{(n)}\right)x \to 0$$

and

$$\lambda_n y + \left(F_1^{(n)} + F_2^{(n)}\right)y \to y,$$
$$\lambda_n y + \left(F_3^{(n)} + F_4^{(n)}\right)y \to 0.$$

Subtracting the first two and the second two and setting $G_n = F_1^{(n)} + F_2^{(n)} - F_3^{(n)} - F_4^{(n)}$ we obtain $G_n x \to x$ and $G_n y \to y$, contradicting the fact that x and y show the failure of "2-density" for $\mathscr{L}$. Since $T \in \operatorname{Alg} \mathscr{L}^{(2)}$ this establishes our observation.

(2) As mentioned earlier, every ABSL with two atoms has the strong rank one density property and the preceding theorem shows that this is false for ABSLs with four atoms. N. Spanoudakis and the first author in [5] give a construction of an ABSL with three atoms also failing the strong rank one density property. The case of finitely many atoms is, therefore, settled.

4. Dual Bases

If X is a reflexive Banach space and $(f_n)_1^\infty$ is an M-basis of X, it is easily show that the sequence $(f_n^*)_1^\infty$, of functionals biorthogonal to $(f_n)_1^\infty$, is an M-basis of X^* with biorthogonal sequence $(\hat{f}_n)_1^\infty$, where $\hat{}$ denotes the canonical mapping of X into X^{**}. It is known [1, Corollary 5.3] that "M-basis" can be replaced by "strong M-basis" in the preceding statement. Concerning the preservation of such properties, note that, whether X is reflexive or not "coming down" from X^* to X causes no problems by virtue of [1, Corollary 5.4]: If $(f_n)_1^\infty$ is a sequence of vectors of X and there is a strong M-basis of X^* which is biorthogonal to it, then $(f_n)_1^\infty$ is a strong M-basis of X. It is easily shown that "strong M-basis" can be replaced by "M-basis" in the latter statement.

Of course, on a non-reflexive space, the sequence $(f_n^*)_1^\infty$ biorthogonal to an M-basis $(f_n)_1^\infty$ may fail to be an M-basis of the dual space for trivial reasons. (For instance, for no M-basis $(f_n)_1^\infty$ of $X = l^1$ can $(f_n^*)_1^\infty$ be an M-basis of $X^* = l^\infty$ since the latter is not separable.) The correct question to ask under these circumstances is: Must $(f_n^*)_1^\infty$ be an M-basis of $\bigvee_{n=1}^\infty f_n^*$? It is easy to show that the answer is affirmative. On the other hand, we give an example below of a strong M-basis $(f_n)_1^\infty$ of $X = c_0$ or c such that $(f_n^*)_1^\infty$ fails to be a strong M-basis of $\bigvee_{n=1}^\infty f_n^*$, even though $\bigvee_{n=1}^\infty f_n^* = X^*$ $(= l^1)$. This example follows from more general theorems that we present.

First we show that if X is any of c_0, c, l^p $(1 \le p < \infty)$ and $(e_n)_1^\infty$ is the usual basis, then the f_n $(n \in \mathbb{Z}^+)$ defined in the proof of Theorem 2.1 are always a strong M-basis, no matter what b_n $(n \in \mathbb{Z}^+)$ we choose (some even zero). We note that the case $p = 2$ of the following theorem is proved in [2].

THEOREM 4.1. *Let X be any one of the Banach spaces c_0, c, l^p $(1 \le p < \infty)$. Let $(e_n)_1^\infty$ be the usual basis of X with biorthogonal sequence denoted by $(e_n^*)_1^\infty$. Let $(b_n)_1^\infty$ be any sequence of scalars and let the sequences $(f_n)_1^\infty, (f_n^*)_1^\infty$ be defined by*

$$f_1 = e_1 + b_1 e_2, \qquad f_{2n-1} = -b_{n-1}e_{2n-2} + e_{2n-1} + b_n e_{2n} \qquad (n \ge 2),$$
$$f_{2n} = e_{2n} \qquad (n \ge 1)$$

and

$$f_{2n-1}^* = e_{2n-1}^* \quad (n \ge 1), \qquad f_{2n}^* = -b_n e_{2n-1}^* + e_{2n}^* + b_n e_{2n+1}^* \quad (n \ge 1).$$

Then $(f_n)_1^\infty$ is a strong M-basis of X with biorthogonal sequence $(f_n^*)_1^\infty$.

Proof. The biorthogonality condition $f_m^*(f_n) = \delta_{mn}$ and the facts that $\bigvee_{n=1}^\infty f_n = X$, $\bigcap_{n=1}^\infty \ker f_n^* = (0)$ are easily verified. Thus $(f_n)_1^\infty$ is an M-basis. To complete the proof we must show (by definition) that $x \in \bigvee\{f_m\colon f_m^*(x) \ne 0\}$, for every element $x \in X$. Let x be given in terms of the basis by $x = \sum_{n=1}^\infty x_n e_n$. Put $J = \{j \in \mathbb{Z}^+\colon f_j^*(x) = 0\}$. At least one of the following three conditions must hold.

(a) J contains an infinite number of odd integers. Let $(n_k)_1^\infty$ be an increasing sequence of positive integers with $\{2n_k - 1\colon k \in \mathbb{Z}^+\} \subseteq J$. Then

$x_{2n_k-1} = e^*_{2n_k-1}(x) = f^*_{2n_k-1}(x) = 0$, for every $k \geq 1$, and so

$$\left\| \sum_{m=1}^{2n_k-1} f^*_m(x) f_m - x \right\| = \left\| b_{n_k} x_{2n_k-1} e_{2n_k} - \sum_{m=2n_k}^{\infty} x_m e_m \right\|$$
$$= \left\| \sum_{m=2n_k}^{\infty} x_m e_m \right\| \to 0, \qquad \text{as } k \to \infty.$$

But, in the sum on the left hand side, we obtain a zero contribution from those terms for which $m \in J$. So in fact we can write

$$\sum_{\substack{m=1 \\ m \notin J}}^{2n_k-1} f^*_m(x) f_m \to x, \qquad \text{as } k \to \infty,$$

showing that $x \in \vee\{f_m\colon m \notin J\}$, as required.

(b) $\mathbb{Z}^+ \setminus J$ contains an infinite number of even integers. Let $(n_k)_1^\infty$ be an increasing sequence of positive integers such that $\{2n_k\colon k \in \mathbb{Z}^+\} \subseteq \mathbb{Z}^+ \setminus J$. Then $f^*_{2n_k}(x) \neq 0$, for every $k \geq 1$, and as

$$\left\| \sum_{m=1}^{2n_k-1} f^*_m(x) f_m + \left(x_{2n_k} - b_{n_k} x_{2n_k-1}\right) f_{2n_k} - x \right\|$$
$$= \left\| \sum_{m=2n_k+1}^{\infty} x_m e_m \right\| \to 0, \qquad \text{as } k \to \infty,$$

arguing as in case (a) we have $x \in \vee\{f_m\colon m \notin J\}$.

(c) For some $k_0 \in \mathbb{Z}^+$ we have $2k \in J$ and $2k - 1 \in \mathbb{Z}^+ \setminus J$, for every $k \geq k_0$. Suppose first that $J = \{2k\colon k \geq k_0\}$. We shall show that, for every non-zero element $y^* \in X^*$ satisfying $y^*(f_m) = 0$, for every $m \notin J$, we have $y^*(x) = 0$. It then follows by the Hahn Banach theorem that $x \in \vee\{f_m\colon m \notin J\}$, as required. Define the sequence $(y_n)_1^\infty$ by $y_n = y^*(e_n)$ $(n \geq 1)$. Then $y^*(f_{2n-1}) = 0$, for every $n \geq 1$, so

$$-b_{n-1} y_{2n-2} + y_{2n-1} + b_n y_{2n} = 0 \qquad (n \geq 1), \tag{1}$$

taking $b_0 = y_0 = 0$. Adding the first N such equations yields

$$b_N y_{2N} = -\sum_{k=1}^{N} y_{2k-1} \qquad (N \geq 1). \tag{2}$$

As $y^*(f_{2n}) = 0$, for $1 \leq n \leq k_0 - 1$, $y_{2n} = 0$, for $1 \leq n \leq k_0 - 1$. From (1) this yields $y_{2n-1} = 0$, for $1 \leq n \leq k_0 - 1$, so $y_n = 0$, for $1 \leq n \leq$

$2k_0 - 2$. Now, as $f_{2n}^*(x) = 0$, for every $n \geq k_0$, we have

$$x_{2n} = b_n(x_{2n-1} - x_{2n+1}) \qquad (n \geq k_0).$$

For every $N \geq k_0$ we have, using (2),

$$\begin{aligned}
\sum_{i=1}^{2N} x_i y_i &= \sum_{i=2k_0-1}^{2N} x_i y_i \qquad (\text{since } y_n = 0, \text{ for } 1 \leq n \leq 2k_0 - 2) \\
&= \sum_{i=k_0}^{N} x_{2i-1} y_{2i-1} + \sum_{i=k_0}^{N} x_{2i} y_{2i} \\
&= \sum_{i=k_0}^{N} x_{2i-1} y_{2i-1} + \sum_{i=k_0}^{N} b_i(x_{2i-1} - x_{2i+1}) y_{2i} \\
&= \sum_{i=k_0}^{N} x_{2i-1} y_{2i-1} - \sum_{i=k_0}^{N} (x_{2i-1} - x_{2i+1}) \left(\sum_{k=1}^{i} y_{2k-1} \right) \\
&= \sum_{i=k_0}^{N} x_{2i-1} y_{2i-1} - \sum_{i=k_0}^{N} (x_{2i-1} - x_{2i+1}) \left(\sum_{k=k_0}^{i} y_{2k-1} \right) \\
&= x_{2N+1} \left(\sum_{i=k_0}^{N} y_{2i-1} \right).
\end{aligned}$$

Hence

$$\sum_{i=1}^{2N} x_i y_i = x_{2N+1} \left(\sum_{i=k_0}^{N} y_{2i-1} \right) \qquad (N \geq k_0). \tag{3}$$

If $X = c_0$ or c, then $x_{2N+1} \to 0$ and, using the facts that $c_0^* = c^* = l^1$, $(\sum_{i=k_0}^{N} y_{2i-1})$ is convergent so bounded. Then $\sum_{i=1}^{2N} x_i y_i \to 0$, so $y^*(x) = \lim_{N\to\infty} \sum_{i=1}^{2N} x_i y_i = 0$. If $X = l^p$ $(1 < p < \infty)$, there is a subsequence of $(\sum_{i=1}^{2N} x_i y_i)$ converging to zero. For, suppose not. Then there exists an integer $k_1 \geq k_0$ and a $\delta > 0$ such that

$$\left| \sum_{i=1}^{2N} x_i y_i \right| \geq \delta \qquad (N \geq k_1).$$

Then, using (3) and Hölder's Inequality,

$$\delta \leq |x_{2N+1}| \left| \sum_{i=k_0}^{N} y_{2i-1} \right| \leq |x_{2N+1}| N^{1/p} \|y^*\|$$

so

$$|x_{2N+1}|^p \geq \frac{\delta^p}{N\|y^*\|^p} \qquad (N \geq k_1).$$

But, since $\Sigma_{N=1}^{\infty} 1/N$ diverges, this contradicts the fact that $\Sigma_{N=1}^{\infty}|x_{2N+1}|^p$ converges. Thus some subsequence of $(\Sigma_{i=1}^{2N} x_i y_i)$ converges to zero and it follows that $y^*(x) = 0$. That $y^*(x) = 0$ in the case where $X = l^1$ is proved similarly.

This completes the proof of the subcase of (c) where $J = \{2k : k \geq k_0\}$. More generally, suppose that $J = \mathscr{E} \cup \{2k : k \geq k_0\}$ where $\mathscr{E}$ is a non-empty subset of $\{1, 2, \ldots, 2k_0 - 2\}$. By what has just been proved,

$$\bigcap_{k=k_0}^{\infty} \ker f_{2k}^* = \vee\{f_n : n \notin \{2k : k \geq k_0\}\}.$$

Hence

$$x \in \bigcap_{m \in J} \ker f_m^* \subseteq \bigcap_{k=k_0}^{\infty} \ker f_{2k}^* = \bigvee_{n \in \mathscr{E}} f_n + \bigvee_{n \notin J} f_n.$$

Thus, $x = y + z$ with $y \in \vee_{n \in \mathscr{E}} f_n$ and $z \in \vee_{n \notin J} f_n$. So $x \in \bigcap_{n \in \mathscr{E}} \ker f_n^*$ and, since $\vee_{n \notin J} f_n \subseteq \bigcap_{n \in \mathscr{E}} \ker f_n^*$, $z \in \bigcap_{n \in \mathscr{E}} \ker f_n^*$. Thus $y \in (\bigcap_{n \in \mathscr{E}} \ker f_n^*) \cap (\vee_{n \in \mathscr{E}} f_n) = (0)$, so $y = 0$ and $x = z \in \vee_{n \notin J} f_n$, as required. This completes the proof of the theorem.

To give an example of a strong M-basis $(f_n)_1^{\infty}$ whose biorthogonal sequence $(f_n^*)_1^{\infty}$ fails to be a strong M-basis of its closed linear span, we shall use the cases where $X = c_0$ or c in the preceding theorem. With notation as in the statement of that theorem, in the case $X = c_0$, $(e_n^*)_1^{\infty}$ is the usual basis of l^1 (identifying c_0^* and l^1 in the usual way). In the case $X = c$ (again identifying c^* and l^1 in the usual way) $(e_n^*)_1^{\infty}$ is the basis of l^1 given by $e_1^* = (1, 0, 0, \ldots)$, $e_2^* = (-1, 1, 0, 0, \ldots)$, $e_3^* = (-1, 0, 1, 0, 0, \ldots)$, $e_4^* = (-1, 0, 0, 1, 0, 0, \ldots), \ldots$. In either case it easily follows that $\vee_{n=1}^{\infty} f_n^* = X^*$, whatever the b_n $(n \in \mathbb{Z}^+)$. We next show that $(f_n^*)_1^{\infty}$ may or may not be a strong M-basis of X^*.

THEOREM 4.2. *Let X be either of the Banach spaces c_0, c and let $(e_n)_1^{\infty}$, $(e_n^*)_1^{\infty}$, $(f_n)_1^{\infty}$, and $(f_n^*)_1^{\infty}$ be defined as in the statement of Theorem* 4.1 *with $(b_n)_1^{\infty}$ any sequence of non-zero scalars. Then $(f_n^*)_1^{\infty}$ is a strong M-basis for $\vee_{n=1}^{\infty} f_n^*$ $(= X^* = l^1)$ if and only if $\Sigma_{n=1}^{\infty} 1/|b_n|$ diverges.*

Proof. Note that, by Theorem 4.1, $(f_n)_1^{\infty}$ is a strong M-basis of X and, by our earlier remarks, $(f_n^*)_1^{\infty}$ is an M-basis of X^*, with biorthogonal sequence $(\hat{f}_n)_1^{\infty}$, whatever the b_n $(n \in \mathbb{Z}^+)$.

Suppose that $\sum_{n=1}^{\infty} 1/|b_n|$ diverges. We show that, for every element $x^* \in X^*$ some subsequence of $((\sum_{k=1}^{2N-1} \hat{f}_k \otimes f_k^*)x^*)$ converges to x^*. It then follows, by [1, Theorem 5.1], that $(f_n^*)_1^{\infty}$ is a strong M-basis of X^*. Let x^* be given in terms of the basis $(e_n^*)_1^{\infty}$ of X^* by $x^* = \sum_{n=1}^{\infty} x_n e_n^*$. Calculation shows that, for every $N \geq 1$,

$$\left(\sum_{k=1}^{2N-1} \hat{f}_k \otimes f_k^* \right) x^* - x^* = b_N x_{2N} e_{2N-1}^* - \sum_{k=2N}^{\infty} x_k e_k^*.$$

As $\sum_{k=2N}^{\infty} x_k e_k^* \to 0$, it is enough to show that some subsequence of $(b_N x_{2N})_1^{\infty}$ converges to zero. If this were false then, for some $\delta > 0$ and $n_0 \in \mathbb{Z}^+$ we have

$$|b_N x_{2N}| \geq \delta \qquad (N \geq n_0),$$

so $|x_{2N}| \geq \delta / |b_N|$, for every $N \geq n_0$. But this contradicts the fact that $(x_n) \in l^1$ (this is obvious in the case $X = c_0$; in the case $X = c$ it is slightly harder to verify). Hence $(f_n^*)_1^{\infty}$ is a strong M-basis of X^*.

Conversely, suppose that $\sum_{n=1}^{\infty} 1/|b_n|$ converges. Define the element $y^* \in X^*$ by $y^* = e_1^* - \sum_{n=1}^{\infty} e_{2n}^*/bn$. Calculation shows that $y^* \in \bigcap_{n=1}^{\infty} \ker \hat{f}_{2n-1}$. However, we show that $y^* \notin \bigvee_{n=1}^{\infty} f_{2n}^*$. It then follows, by [1, Theorem 5.1] (since $\bigcap_{n=1}^{\infty} \ker \hat{f}_{2n-1} \neq \bigvee_{n=1}^{\infty} f_{2n}^*$), that $(f_n^*)_1^{\infty}$ is not a strong M-basis of X^*.

In either of the cases $X = c_0$ or c identify y^* and f_{2n}^* $(n \in \mathbb{Z}^+)$ with their images in l^1 under the usual linear isometry of X^* onto l^1. In the case $X = c_0$ we have $y^* = (1, -1/b_1, 0, -1/b_2, 0, \ldots)$ and $f_{2n}^* = (0, 0, \ldots, 0, -b_n, 1, b_n, 0, 0, \ldots)$, where 1 occurs as the $2n$th term. The element $u^{**} = (1, 0, 1, 0, 1, 0, \ldots) \in l^{\infty} = (l^1)^*$ satisfies $u^{**}(f_{2n}^*) = 0$, for every $n \in \mathbb{Z}^+$ and $u^{**}(y^*) = 1$. Thus $y^* \notin \bigvee_{n=1}^{\infty} f_{2n}^*$. In the case $X = c$ we have

$$y^* = \left(1 + \sum_{k=1}^{\infty} \frac{1}{b_k}, -\frac{1}{b_1}, 0, -\frac{1}{b_2}, 0, -\frac{1}{b_3}, 0, \ldots \right)$$

and $f_2^* = (-1 - 2b_1, 1, b_1, 0, 0, \ldots)$, $f_{2n}^* = (-1, 0, 0, \ldots, 0, -b_n, 1, b_n, 0, 0, \ldots)$ $(n \geq 2)$, where 1 occurs as the $2n$th term. In this case the element $v^{**} = (1, 1, 2, 1, 2, 1, 2, \ldots) \in l^{\infty}$ satisfies $v^{**}(f_{2n}^*) = 0$, for every $n \in \mathbb{Z}^+$ and $v^{**}(y^*) = 1$. Hence $y^* \notin \bigvee_{n=1}^{\infty} f_{2n}^*$ once again. This completes the proof.

Observe that, in the one remaining case, $X = l^p$ $(1 < p < \infty)$, where the strong M-basis of X defined in the statement of Theorem 4.1 has biorthogonal sequence $(f_n^*)_1^{\infty}$ satisfying $\bigvee_{n=1}^{\infty} f_n^* = X^*$, $(f_n^*)_1^{\infty}$ is indeed a strong M-basis of X^*. This follows from our earlier remark since X is reflexive.

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